Lecture notes on risk management, public policy, and the financial system

Portfolio credit risk models

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Standard model for portfolios: overview

- Correlations of two individual firms' asset returns key parameters of their default correlation
- Assume no correlation between idiosyncratic risk of different firms
- (Eventually,) assume all obligors identical:
  - Same default probability for all credits
  - Same default correlation for all pairs of credits
- Exploit **conditional independence**: once a realization of the market factor is stipulated, firms' returns independent
- Law of Large Numbers⇒idiosyncratic risk disappears
- Model distribution of portfolio credit *loss* as if it were probability distribution of single-obligor *default*
  - Correlation nonetheless affects default distribution, in conjunction with market shock

Asset return correlation in the single-factor model

- Firms $i = 1, 2, \ldots$, each with its own β_i to the market factor m and its own standard normal idiosyncratic shock ϵ_i:

$$r_i = \beta_i m + \sqrt{1 - \beta_i^2}\epsilon_i, \qquad i = 1, 2, \ldots$$

- β_i is correlation of firm i's return to *market* return
- Assume no correlation between idiosyncratic risk of different firms: ϵ_i uncorrelated across firms:

$$\mathbf{E}\left[\epsilon_i \epsilon_j\right] = 0, \qquad i, j = 1, 2, \ldots$$

- $\Rightarrow$Asset returns of firms i and j follow bivariate standard normal distribution
  - Mean of each firm's return is 0, variance of each firm's return is 1
  - Asset return correlation of firms i and j is $\beta_i \beta_j$
  - **Example**: $\beta_i = 0.25, \beta_j = 0.5 \Rightarrow$ asset return correlation 0.125

Asset return and default correlation

- Return correlation related, but not identical, to default correlation
  Asset return correlation: $\beta_i \beta_j$
  Default correlation ρ_{ij} related to asset return correlation $\beta_i \beta_j$ by

$$\rho_{ij} = \frac{\pi_{ij} - \pi_i \pi_j}{\sqrt{\pi_i(1-\pi_i)}\sqrt{\pi_j(1-\pi_j)}}$$

  Joint distribution of asset returns of ith, jth firms $\Phi(r_i, r_j; \beta_i \beta_j)$
  - Joint CDF of two standard normal variates with a correlation of ρ_{ij}

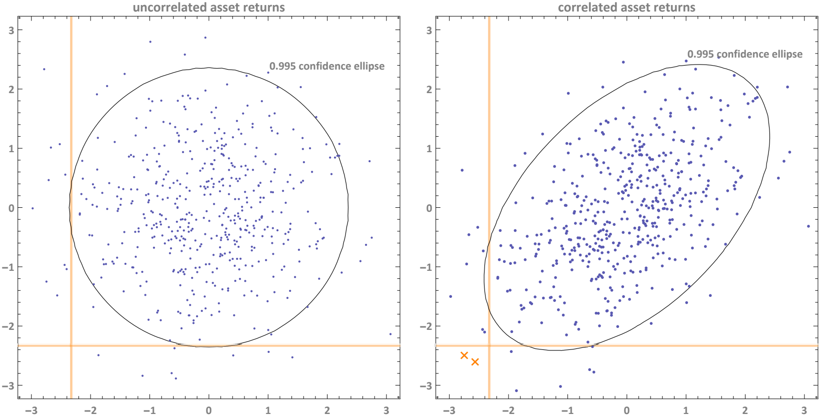
  Joint default probability of ith, jth firms is

$$\pi_{ij} = \Phi(k_i, k_j; \beta_i \beta_j)$$

  - k_i, k_j: firm i and j default thresholds
- Asset return correlation and default correlation thus related by

$$\boxed{\Phi(k_i, k_j; \beta_i \beta_j) = \pi_i \pi_j + \rho_{ij}\sqrt{\pi_i(1-\pi_i)}\sqrt{\pi_j(1-\pi_j)}}$$

Correlated and uncorrelated defaults



Simulation of defaults applying the single-factor model in a portfolio of two credits, both with $\pi = 0.01$. Left panel: correlation coefficient $\rho = 0$. Right panel: correlation coefficient $\rho = 0.50$. Orange grid lines are placed at default thresholds. Simulated return pairs marked by points if they result in default of at most one credit and by x's if they result in default for both. Realizations of the asset return pair have a 99.5 percent probability of falling within the density contour.

Asset return and default correlation: example

- Identical firms with common default threshold k and probability $\pi = 0.01$
- Asset return correlation and default correlation related by

$$\Phi(k, k; \beta^2) = \pi^2 + \rho\pi(1 - \pi)$$

- Use relationship to
  - Assume value for default correlation and solve joint default probability $\Phi(k, k; \beta^2)$ for asset correlation β^2
  - Assume value for β and calculate default correlation ρ via $\Phi(k, k; \beta^2)$

Market return correlation β	0.5251	$\sqrt{0.25}$
Asset return correlation β^2	0.2757	0.25
Default correlation	0.04	0.0341
Joint default probability	4.9600×10^{-4}	4.3752×10^{-4}

Default correlation in the single-factor model

Portfolio credit VaR in the single-factor model

- Derivation of the credit loss distribution function
- Portfolio credit loss distribution
- Portfolio credit VaR

From conditional default probability to portfolio loss

- Additional assumptions on credit portfolio:
  - Identical obligors: market risk factor loading β, pairwise correlation β^2, default probability $\pi = \Phi(k)$
  - Granularity: homogeneous and completely diversified portfolio
  - Zero recovery
- ⇒Conditional default probability common to all obligors:

$$p(m) = \Phi\left(\frac{k - \beta m}{\sqrt{1-\beta^2}}\right) = \Phi\left(\frac{\Phi^{-1}(\pi) - \beta m}{\sqrt{1-\beta^2}}\right) \qquad \forall i = 1, 2, \ldots$$

- Law of Large Numbers⇒
  - Granularity⇒idiosyncratic risk disappears
  - Portfolio loss a function *only* of market shock
- Fraction x of loans defaulting—portfolio *loss rate*—equals single-firm default *probability*, conditional on market shock:

$$x = p(m) = \Phi\left(\frac{\Phi^{-1}(\pi) - \beta m}{\sqrt{1-\beta^2}}\right)$$

Probability distribution of the credit loss rate

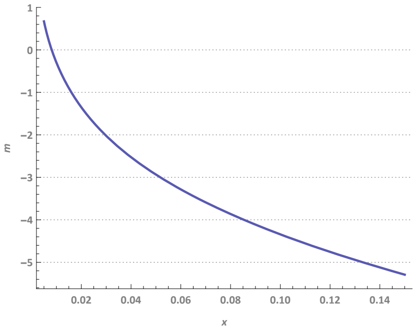
- Loss rate $x = p(m)$ is random, because it is a function of latent random factor, market shock m
- What is probability distribution of x?
- We've posited a standard normal distribution for m, from which we can derive distribution of x
  1. Find market shock m that leads to a given loss rate x
  2. Probability of loss rate x equals probability of market shock m that leads to it

Market factor and loss rate

- Step 1: solve for m as a function of x:

$$m = \frac{\Phi^{-1}(\pi) - \sqrt{1-\beta^2}\Phi^{-1}(x)}{\beta}$$

- Sharply negative market factor m corresponds to high loss rate x



Market factor as a function of loss rate. Default probability $\pi = 0.01$ (1 percent, $k = -2.33$), $\beta = 0.25$.

Credit loss distribution

- Step 2: associate probability of loss rate x with that of corresponding market shock m
- Recall m a standard normal variate:

$$\mathbf{P}\left[\tilde{m} \leq m\right] = \Phi\left[m\right]$$

- $\Rightarrow$Cumulative probability distribution function of credit loss is

$$\mathbf{P}\left[\tilde{x} \leq x\right] = \mathbf{P}\left[\tilde{m} \geq m\right] = 1 - \mathbf{P}\left[\tilde{m} \leq m\right] = 1 - \Phi\left[m\right] = \Phi\left[-m\right]$$

- Therefore

$$\boxed{\mathbf{P}\left[\tilde{x} \leq x\right] = \Phi\left[\frac{\sqrt{1-\beta^2}\Phi^{-1}(x) - \Phi^{-1}(\pi)}{\beta}\right]}$$

  - The complicated term "inside" is the market factor realization m corresponding to any given loss rate x
  - And m is a standard normal, the standard normal CDF "outside" is that of m

Market factor and portfolio loss distribution

- The probability of a loss in excess of any stipulated level x is then

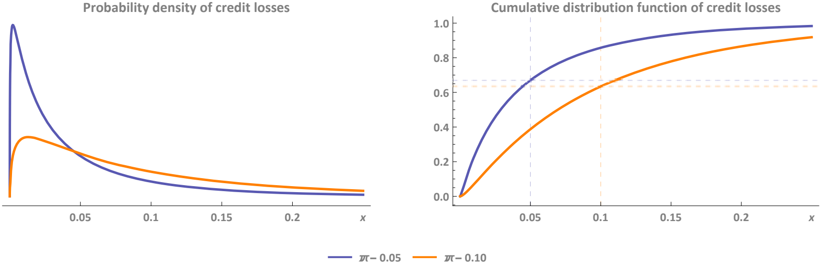
$$\mathbf{P}\left[\tilde{x} \geq x\right] = 1 - \mathbf{P}\left[\tilde{x} \leq x\right] = \mathbf{P}\left[\tilde{m} \leq \frac{\Phi^{-1}(\pi) - \sqrt{1-\beta^2}\Phi^{-1}(x)}{\beta}\right]$$
$$= \Phi\left[\frac{\Phi^{-1}(\pi) - \sqrt{1-\beta^2}\Phi^{-1}(x)}{\beta}\right]$$

  - A *high* loss rate x corresponds to a market factor realization with a *low* probability
  - Probability of realizing a loss rate no higher than x is therefore *high*
- Random loss rate $\tilde{x}$ *below* level $x \Leftrightarrow$ realized value of market factor $\tilde{m}$ *higher* than associated level m

Impact of default probability

- For realistic default probabilities below 50 percent, median portfolio loss rate is below the loan default rate
- Low default probability:
  - For moderate correlation, low default probability induces Bernoulli-like, “binary” loss behavior in the portfolio
  - Loss density very skewed to low loss levels
  - High likelihood that portfolio losses low
- High default probability:
  - Higher likelihood of higher portfolio losses
  - Loss density more spread out over range of loss levels

Single-factor model portfolio loss distribution



Granular portfolio, $\beta = \sqrt{0.3} = 0.5477$ for all obligors. Losses expressed as a fraction of portfolio par value.

Market factor and portfolio loss distribution

- Although treating portfolio "as if" a single credit, correlation to market factor β still affects default distribution
  - Correlation operates through market shock
- Expected loss (EL) rate equals typical portfolio constituent's default probability π, constant across the many small obligors
- For $\pi \leq 0.5$ (typical default rates),

$$\mathbf{P}[\tilde{x} \leq \pi] > 0.5$$

and

$$\lim_{\beta \to 0} \mathbf{P}[\tilde{x} \leq \pi] = 0.5$$

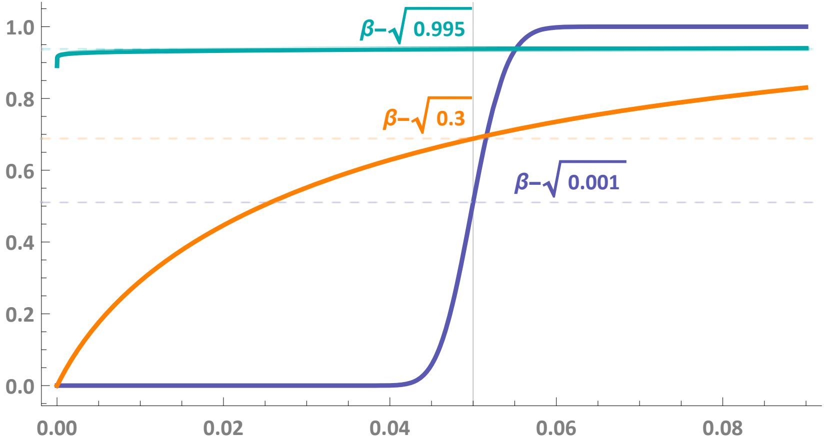
  - Median portfolio loss rate below default probability π when correlation moderate
  - Correlation benefit: probability that portfolio loss below typical portfolio constituent's default probability greater than 50%
  - Median loss close to default probability π when correlation low

Impact of correlation on credit loss distribution

- Correlation near 1: portfolio behaves as if single loan/obligor
  - Loss distribution close to binary
    - $\mathbf{P}\left[\tilde{x} \leq \varepsilon\right]$ (nearly no loss) near $1 - \pi$
    - $\mathbf{P}\left[\tilde{x} \leq 1 - \varepsilon\right]$ (near-complete loss) near π

    with ε a tiny positive number
  - Low probabilities of intermediate outcomes
  - Intuition: With high correlation, default clusters very likely
- Correlation near 0
  - High probability of portfolio loss rate very close to typical firm's default probability
  - Intuition: With low default rates and low correlation, default clusters close to impossible
- Correlation "in the middle"
  - Intuition: with low default rates and intermediate correlation, default clusters are unusual

Single-factor model: correlation and loss distribution

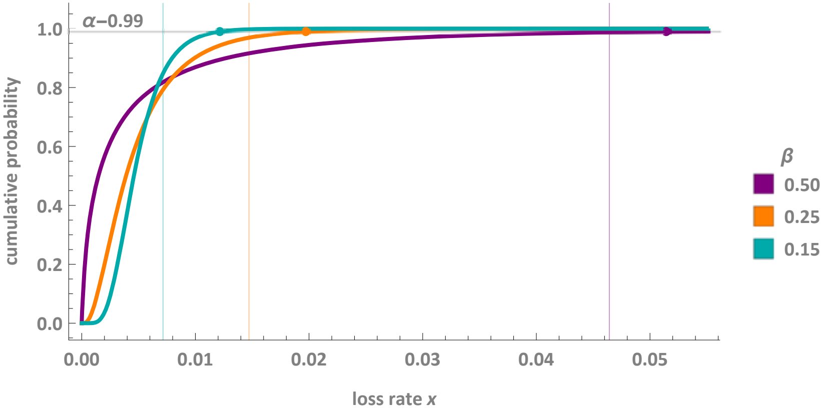


Granular portfolio; default probability 5 percent. Losses expressed as a fraction of portfolio par value.

Portfolio credit VaR

- Loss distribution function→quantiles of $\mathbf{P}[\tilde{x} \leq x]$
- Quantiles of $\mathbf{P}[\tilde{x} \leq x]$ (minus EL) →credit VaR
- Higher correlation leads to higher VaR
  - By increasing likelihood of default clusters

Portfolio credit VaR in the single-factor model



Granular portfolio; default probability 0.5 percent. Losses expressed as a rate or fraction of portfolio par value. Color-coded vertical grid lines indicate credit VaR at 99-percent confidence level for each default correlation assumption. Color-coded points mark quantiles of portfolio credit losses for each default correlation assumption.